$\widetilde{R}_i$ = i^{th} element of $\widetilde{R}^{(n)}$
$[R]$ = matrix of $R^{(n)}$
$[\widetilde{R}]$ = matrix of $\widetilde{R}^{(n)}$
S_{ji} = j, i^{th} element of $[S]$
$[S]$ = Gramm–Schmidt orthonormalization matrix
V_z = axial velocity
V = maximum velocity in Poiseuille profile
x = Dz/VR^2 = axial coordinate, dimensionless
y = r/R = radial coordinate, dimensionless
z = axial coordinate

Greek Letters

α_{ij} = i, j^{th} element of $[A]$
β_{ij} = i, j^{th} element of $[B]$
δ_{ij} = Kronecker delta
ϵ = error
λ_n = n^{th} eigenvalue
ϕ_i = i^{th} trial function
ψ_n = n^{th} eigenfunction
ω_i = i^{th} solution of Equation (13)

Superscript

T = transpose

LITERATURE CITED

1. Brown, G. M., *A.I.Ch.E. J.*, **6**, 179 (1960).
2. Sellars, J. R., Myron Tribus, and J. S. Klein, *Trans. Am. Soc. Mech. Engrs.*, **78**, 441 (1956).
3. Lauwerier, H. A., *Appl. Sci. Res.*, **A8**, 366 (1959).
4. Hsu, C. J., *A.I.Ch.E. J.*, **11**, 939 (1965).
5. Cleland, F. A., and R. H. Wilhelm, *ibid.*, **2**, 489 (1956).
6. Katz, S., *Chem. Eng. Sci.*, **10**, 202 (1959).
7. Walker, R., *Phys. Fluids*, **4**, 1211 (1961).
8. Krongelb, S., and M. W. P. Strandberg, *J. Chem. Phys.*, **31**, 1196 (1959).
9. Kantorovich, L. V., and V. I. Krylov, "Approximate Methods of Higher Analysis," P. Noordhoff, Groningen, Netherlands (1964).
10. Duncan, W. J., *Aeronaut. Res. Comm. Rept. Memo. No. 1798* (Aug. 3, 1937); *Tech. Rept. Aeronaut. Res. Comm.*, **1**, 484-516 (1937).
11. Carslaw, H. S., and J. C. Jaeger, "The Conduction of Heat in Solids," p. 491, Oxford Univ. Press, London (1959).
12. Newman, A. B., and L. Green, *Trans. Electrochem. Soc.*, **66**, 355 (1934).
13. Digital Computer Lab., Univ. Illinois, Urbana, Library Subroutine F2-UOI-EIGI-29-FR (1962).

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External Flows of Viscoelastic Materials: Fluid Property Restrictions on the Use of Velocity-Sensitive Probes

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The more pronounced macroscopic features of flows of viscoelastic materials around submerged objects are considered in the light of restrictions imposed on the flow by the Deborah number, a dimensionless group representing a ratio of time scales of the fluid and the flow process. It is seen that one major effect is to thicken the boundary layer appreciably in the region of the leading edge or stagnation point of the object in the fluid; depending on the shape of the object this thickening of the boundary layer may be felt for appreciable distances into the velocity field.

The influence of these and other effects on the use of probes for determination of point values of the velocity of viscoelastic fluids is considered in some detail. The general effect is to impose strong restrictions on the use of such probes and on the interpretation of measurements made with them. These predicted limitations are supported, in a general way, by recent experimental measurements; thus the present macroscopic analysis appears to define several major effects to be considered in subsequent, more detailed investigations.

Some general features of flows external to objects of various geometries and of boundary-layer flows appear to differ grossly between viscoelastic and Newtonian fluids; the purpose of the present paper is to describe these features macroscopically, for ordering purposes, and to make a start on the detailed analysis of the more important of these.

While external flows may be broadly of interest in engineering, the analysis presented is directed particularly toward an understanding of flows around objects inserted in the fluid to sense its local velocity or pressure, such as heated cones or wires, impact tubes, and small bubbles or particles. It will be seen that the deformational behavior

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of viscoelastic fluids introduces important restrictions on the use of such probes, and major changes in the interpretation of their readings may be necessary.

ANALYSIS

Hot-Wire or Hot-Film Probes

Effects Within the Boundary Layer. Consider flow over a wedge or cone of the geometry shown in Figure 1. As a fluid element approaches the object and crosses the dashed line indicating the hypothetical edge of the boundary layer, it is subjected to deformation rates which change rapidly with position downstream or, from the viewpoint of an observer moving with the fluid element, rapidly with time. In fact, if one considers elements of fluid which cross into the purely viscous boundary-layer region near the leading edge of the object, the change in deformation rate of the fluid elements considered may be nearly discontinuous. Such deformational processes may be characterized by large values of the Deborah number (*2, 22, 23*), defined as

$$N_{Deb} = \theta_{fl} \sqrt{D\Gamma/Dt} \tag{1}$$

Here θ_{fl} denotes the relevant relaxation time of the fluid, Γ is the square root of the second invariant of the rate of strain tensor (here taken as equivalent to the shear rate), and D/Dt is the material or the Oldroyd derivative, the distinction being immaterial for scalar quantities such as Γ. For materials exhibiting a spectrum of relaxation times, rather than merely a single value, θ_{fl} may be taken as the maximum or limiting value encountered in the experiment under consideration.

It may be shown analytically (*2, 21* to *23, 26*) that large values of the Deborah number imply a solidlike response by the material being deformed, while small Deborah numbers imply fluidlike behavior. This conclusion arises rather generally in all descriptions of nonlinear viscoelastic behavior which are encompassed by simple fluid theory. Experimentally, dilute polymeric solutions may be used to demonstrate these differing asymptotic responses dramatically. In the case of steady laminar shearing flows ($N_{Deb} = 0$) the material response may be fluidlike with a viscosity level of only a few centipoises. This asymptotic behavior of dilute solutions is, of course, well known and requires no further documentation. At the other asymptotic extreme, if the same dilute solution is stressed suddenly, or nearly discontinuously, ($N_{Deb} \to \infty$) as by a sudden impact with a blunt object, it deforms as a sheet of material which exhibits strong elastic recoil and may return essentially to its initial configuration with little or no evidence of any flow or other fluidlike response. This latter experiment may be carried out conveniently by employing a hammer and a few milliliters of the solution; the time scale of the experiment is of the order of a tenth of a second with typical solutions, as determined from high-speed motion picture studies of the phenomenon (*20*). It is important to emphasize in connection with this discussion, as there has been some confusion in the literature concerning this point, that the deformation rates to which the material is subjected play no essential role in determining whether one asymptotic condition or the other is being approached. The Deborah number as defined by Equation (1) is zero in all steady laminar shearing flows, regardless of the level of the actual deformation rates encountered, and a fluidlike response is, of course, observed at all deformation rate levels in such experiments. Similarly, at the other asymptotic extreme, it is the magnitude of $D\Gamma/Dt$ and not of Γ which determines the magnitude of the Deborah number.

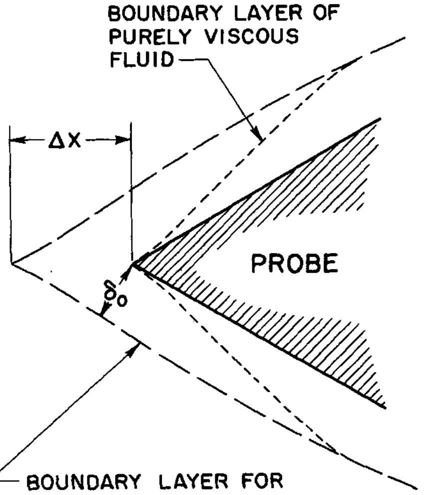

Fig. 1. Schematic diagram of differences between viscoelastic and purely viscous boundary layers developing on a solid surface (probe).

To return to the boundary-layer problem of interest, if an element of the viscoelastic materials entered the boundary-layer region near the leading edge of the object, as defined by the purely viscous fluid curves of Figure 1, at high external fluid field velocities, a Deborah number of sufficient magnitude to imply a solidlike material response might be anticipated. Superficially such considerations would imply the development of a blob of solidlike stagnant material coating the region surrounding the stagnation point. If this were to occur, the stagnant material would then have time to relax from its sudden deformation changes and thus no longer exhibit solidlike properties, and some intermediate condition would be attained. These considerations imply a region surrounding the forward stagnation point in which the fluid velocities, though not zero, are in fact markedly below their free-stream values (*21*). These are just the flow conditions within a well-developed boundary layer, and as pointed out by Astarita (*2*), such considerations are equivalent to assuming the boundary layer to begin somewhat upstream of the stagnation point or leading edge of the object, as indicated by the viscoelastic fluid curves in Figure 1. Further downstream the viscoelastic fluid relaxes from these leading edge effects and the boundary layer may become identical to that for purely viscous fluids, unless normal stress effects of the kind considered in earlier analyses (*7, 38*) are still important. For the latter to be the case, however, unusual fluids or flows would appear to be required to produce Weissenberg numbers, representing the ratio of elastic to viscous forces developed in the material, of a sufficient magnitude to enable the development of measurable influences (*38*).*

In order to determine the thickness of the partially developed viscoelastic boundary layer at the leading edge of the object (δ_o in Figure 1) let us consider the Deborah number as defined by Equation (1) in more detail. In steady flows the material derivative may be approximated by

$$\frac{D\Gamma}{Dt} \approx U \frac{\Delta\Gamma}{\Delta x} \tag{2}$$

$$\approx \frac{U\left(\frac{U}{\delta_o} - 0\right)}{\Delta x} \tag{2a}$$

* Note added in proof: The reader may wish to refer to the recent boundary-layer study of Hermes and Fredrickson (*A.I.Ch.E. J.*, 13, 253 (1967), which compliments the present considerations.

in which U denotes the free-stream velocity, δ_o is the boundary-layer thickness at the leading edge of the object, and Δx is the distance upstream to which the boundary layer projects. As a very rough first approximation let us also assume that values of Δx appreciably greater than δ_o represent unstable flow patterns; this enables one to equate δ_o and Δx at least for purposes of order-of-magnitude arguments. Then if the maximal value of the Deborah number is taken to be of the order of unity, as is the case in several simpler velocity fields which may be analyzed more rigorously (*2, 22*), Equations (1) and (2) yield

$$\delta_o \approx \Delta x \approx \theta_{fl}\, U \tag{3}$$

As the relaxation times of dilute polymeric solutions of primary interest in turbulent flow fields (concentrations ranging from 100 p.p.m. to 1%) are typically in the range of 10^{-2} to 10^{-4} sec. (*25, 30, 33*), at a free-stream velocity of 10 ft./sec. the predicted values of δ_o are in the range of 10^{-1} to 10^{-3} ft. For hot-wire or hot-film probes having maximum dimensions of 0.001 to 0.01 in., this corresponds to film thicknesses comparable to or even larger than the size of the probe itself. Obviously the probe response would be expected to be very sluggish with such fluids unless the free-stream velocity is reduced to decrease δ_o appreciably, or unless the probe is so large that the finite value of this boundary-layer thickness at the forward stagnation point (or leading edge) is small as compared with that elsewhere on the probe. The latter alternative would not, however, resolve the problem of the probe being covered by a relatively thick boundary layer everywhere. Therefore its response would be expected to be sluggish and the probe useful only for measurement of time-averaged quantities.

An even more interesting conclusion arises from a consideration of boundary-layer thickness δ at a distance x from the leading edge of the probe. In the case of Newtonian fluids flowing over a surface which may be approximated by a flat plate, the laminar boundary-layer thickness is given as

$$\delta = a_o \left(\frac{x\nu}{U} \right)^{1/2} \tag{4}$$

while the heat transfer coefficient h, being inversely proportional to δ, is given as (*8*)

$$\frac{h}{k} = \frac{1.539}{a_o} (N_{Pr})^{1/3} \sqrt{\frac{U}{\nu x}} \tag{5}$$

For viscoelastic liquids, the total boundary-layer thickness, as a first approximation, may be taken as that thickness which would be developed if the boundary layer began a distance $U\,\theta_{fl}$ forward of the leading edge:

$$\delta \approx a \left[(x + U\,\theta_{fl}) \frac{\nu}{U} \right]^{1/2} \tag{4a}$$

In calculating the heat transfer coefficient the consistent assumption yields

$$\frac{h}{k} = \frac{1.539}{a} (N_{Pr})^{1/3} \left(\frac{U}{\nu(x + U\,\theta_{fl})} \right)^{1/2} \tag{5a}$$

This implies the velocity profiles to be similar at all positions in the boundary layer, beginning at the actual leading edge of the object. This question is considered more fully elsewhere (*2*) and while not correct in detail appears to be a realistic first approximation.

At low fluid velocities or far downstream, $x >> U\,\theta_{fl}$, and the heat transfer coefficient given by Equation (5*a*) is just the Newtonian value, Equation (5). As indicated earlier, this is in agreement with other analyses for this geometry. Conversely, however, for highly elastic fluids or at high velocities, $U\,\theta_{fl} >> x$, and for a given probe (x fixed) one obtains

$$\frac{h}{k} = \frac{1.539}{a} (N_{Pr})^{1/3} (\nu\theta_{fl})^{-1/2} \tag{6}$$

that is, the heat transfer coefficient becomes independent of fluid velocity. In other words the utility of heated probes as velocity measuring devices is predicted to be limited to velocities such that the Deborah number based upon x, the distance from the leading edge of the probe, is much less than unity; that is

$$N_{Deb,x} = \frac{\theta_{fl} U}{x} << 1.0 \tag{7}$$

In addition the usual limitation that the Grashof-Prandtl number product exceed 10^{-4} (*14*) to render effects of natural convection negligible is extremely difficult to meet in liquids; thus in practice the Reynolds number must be quite large to enable one to minimize, at least, these effects. Unfortunately, the probe requirements to counter these two effects are conflicting: the probe should be as small as possible to minimize the Grashof number (natural convection) but large to minimize the Deborah number (viscoelastic effects on the boundary layer of the probe). Correspondingly a restriction to low velocities meets the Deborah number criterion of Equation (7) but aggravates the natural convection limitations.

Effects in the Velocity Field Outside the Boundary Layer. Consider the potential velocity distribution in the case of flow normal to a long cylinder of diameter d, which is given by (*5*):

$$v = U_o \sqrt{1 + \frac{d^4}{16\,r^4} - \frac{d^2}{2\,r^2} \cos(2\theta)} \tag{8}$$

in which U_o denotes the unperturbed free-stream velocity, r is the distance from the tube axis, θ is the angle from the stagnation line, and v is the magnitude of the velocity vector.

The stretching rate Γ_s, that is, the rate of change of v along a streamline, decreases very rapidly with increasing r, so that only the region in which $r \cong d/2$ will be considered as a first approximation. In this region

$$\Gamma_s \cong \frac{2}{d} \frac{\partial v}{\partial \theta} \cong \frac{4\,U_o}{d} \cos\theta \tag{9}$$

and, while a maximum at the stagnation point, remains of the same order of magnitude over a major portion of the cylinder.

The stress developed in the steady stretching of a flat sheet of viscoelastic material described by means of the convected Maxwell constitutive equation* having a single relaxation time θ_{fl} (*13, 39*) is readily shown to be given by

$$\Delta\tau = \frac{4\,\mu\,\Gamma_s}{1 - (2\theta_{fl}\Gamma_s)^2} \tag{10}$$

* It is not the purpose of this paper to compare the relative merits of a variety of constitutive equations useful in describing viscoelastic materials; such a compilation has recently been prepared (*34*). The present choice is supported by the desire to portray at least semi-quantitatively all of the effects likely to arise in real fluids in the simplest manner. Several papers attest to the utility of the chosen formulation (*12, 13, 15, 36, 39*). Furthermore, alternate choices may be shown to change the numerical coefficients but not the general conclusions to be drawn. Thus, the use of a full n^{th} order Rivlin-Ericksen expansion as a proper approximation to the behavior of simple fluids yields a result of the same form, although there appears to be no independent method for determining the magnitude of the material physical property coefficients in this latter case.

in which μ denotes the viscosity of the fluid and $\Delta\tau$ is the excess normal stress traction along the streamlines. The development of Equation (10) is a simple one and need not be reproduced in detail; the reader may refer to Lodge (*17*) for details. One sees from Equation (10) that the maximum possible stretching rates are limited by the requirement that the stress remain finite, say

$$\Gamma_{max} = \frac{1}{2\theta_{fl}} \tag{11}$$

Combining Equations (9) and (11) one obtains

$$\left(\frac{U_o}{d}\right)_{max} = \frac{1}{8\,\theta_{fl}\cos\theta} \tag{12}$$

The significance of Equation (12) is that it fixes the maximum value of the free-stream velocity at which, for a given cylinder diameter, potential flow may be maintained external to a conventional boundary layer. If this maximum velocity is exceeded, the external flow will depart from that of a potential velocity field in such a direction as to maintain the fluid stretch rate below the maximum level given by Equation (11). This effect is thus also manifested essentially as a thickening of the boundary layer through an increase in the effective radius of curvature of the external velocity field in which a potential flow distribution may be assumed.

Choosing values of $\cos\theta$ and of θ_{fl} as 0.8 and 10^{-3} sec. as typical of the forward region of the cylinder and the fluids of interest, respectively, and a free-stream velocity of 1 ft./sec., one obtains from Equation (12), for the minimum value of tube diameter which is able to support a potential velocity distribution external to a boundary layer of conventional thickness

$$d_{min} = 8\,U_o\,\theta_{fl}\cos\theta$$

$$\approx 0.08 \text{ in.} \tag{13}$$

That is to say if a cylinder has a diameter smaller than indicated under the flow conditions assumed, the external potential velocity field will adjust itself to the radius of curvature calculated (0.04 in.) rather than to the radius of the cylinder itself. Under more extreme conditions the velocity field near the cylinder would thus appear to become independent of the diameter of the cylinder, taking on a curvature defined instead by the calculated value of d_{min} as given by Equation (13).

The limitation imposed by the external field [Equation (13)] on use of heated cylindrical probes as velocity-sensitive devices is similar to that imposed by boundary-layer considerations [Equations (5*a*) and (7)] in the sense that in both cases the maximum velocity levels at which useful measurements may be made are predicted to be severely reduced below the levels of interest; again the problem may be resolved through use of larger probes if the physical and time scales of the experiment permit such large probes. In both cases it is predicted that, with a given probe, there will be some cutoff velocity level above which the sensitivity of the probe to further increases in velocity is greatly reduced.

Comparison with Experiment. Clearly the arguments used to derive the equations in the previous sections are only first approximations. They suffice, however, to indicate that flows about submerged objects may differ grossly between Newtonian and viscoelastic fluids. It is therefore of interest to inquire whether or not the general effects predicted are supported experimentally, in which case the more careful analyses necessary to describe the velocity fields in detail would be warranted and worthwhile.

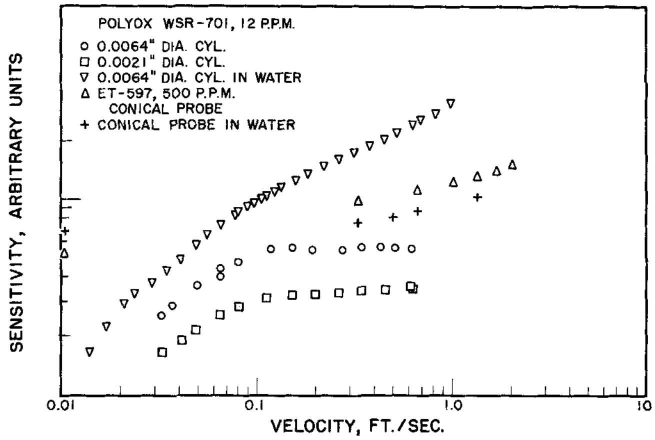

Fig. 2. Comparison of heat transfer rates to Newtonian and viscoelastic fluids. Data of Acosta and James (*1*) and of Leathrum (*16*). Data of reference *1* obtained by towing the probe through quiescent fluid, and those of reference *16* obtained by inserting the probe into a turbulent ducted field (2.47-in. diameter tube).

Heat transfer measurements have been made by Acosta and James (*1*) and by Leathrum (*16*), the former using small cylindrical probes and the latter a coated cylindrical cone. Both sets of probes and the hardware associated with them were purchased from Thermo Systems, Inc., Minneapolis. Water and dilute polymeric solutions were used as test fluids. Typical sets of data are reproduced in Figure 2.

When one considers first the data for cylindrical probes, the sharpness with which the transition to a boundary layer which is uninfluenced by the external flow field—hence to a heat transfer coefficient which is independent of velocity—occurs is remarkable; such velocity-insensitive regions were encountered by Acosta and James over wide ranges of concentration level of the polymeric solutions used.

The fact that the boundary-layer thickness, hence the heat transfer coefficient, becomes independent of velocity at higher velocity levels is in agreement with the predictions of Equations (6*a*) and (13). Furthermore, approximate values of the fluid relaxation time θ_{fl}, as computed from Equations (6*a*) or (13), and this critical velocity are in at least approximate agreement with the values expected on the basis of measurements on similar, although not identical, systems (*25, 30, 33*). These points of agreement are gratifying and support the general considerations used in derivation of Equations (6*a*) and (13). However, the experimental data of Acosta and James also show an essential independence of the critical velocity on probe diameter, under all conditions used. This would appear to be reasonable only if the radius of curvature of the velocity field outside the boundary layer, Equation (13), becomes essentially infinite (in comparison with the probe diameter) at this point. Whether this is the case or whether the analysis is too crude to portray an independence on probe size, for cylindrical probes, remains to be resolved.

Turning to the conical probe data of Figure 2, these were obtained by using a probe with a diameter of 0.010 in. at the heated section. The water-soluble polyacrylamide used (ET 597) has a time constant of no less than 2×10^{-4} sec. (*25*). Although the curves are very flat, reflecting the high level of the transport rates by natural convection, no break comparable to those in the Acosta-James data may be found. This is reasonable in view of the much larger size of the Leathrum probe. The heated area was located approximately 0.02 in. from the tip of the cone;

thus the maximum value of $U\,\theta/x$ achieved [Equations (5a) and (7)] is only of the magnitude of 0.2.

Finally, it may be noted that under conditions of high Deborah number the large boundary-layer thicknesses and low fluid velocities near the surface of the probe implied by Equations (6) and (13) suggest that the probe may be unusually sensitive to disturbances due to natural convection even though the Reynolds number of the system as a whole is fairly large. Merrill (*18*) has noted the calibration curves to be at least double valued in viscoelastic fluids at high fluid velocities; whether this is due to instabilities caused by temperature gradients in the thick boundary layer is not known for certain but is evidently a clear possibility.

IMPACT TUBES

In the case of viscoelastic fluids three possible effects may serve to limit or at least to modify the utility of Pitot or similar impact probes, as a matter of principle. In addition the response of small probes in fluids having the viscosity levels of interest may be sluggish, and this fact makes these techniques in general somewhat less attractive than in the case of aerodynamic measurements. The limitations in principle are:

1. Under laminar flow conditions the reading is dependent on the magnitudes of the deviatoric normal stress terms, as well as on the fluid momentum. Using Δp to denote the difference between an impact pressure and the wall value of the hydrostatic pressure, one obtains

$$\Delta p = -(\tau_{11} - \tau_{22}) - \int_r^R (\tau_{22} - \tau_{33})\, d\ln r + \frac{\rho\, u_1^2}{2} \quad (14)$$

in which the τ_{ii} terms represent the total normal stress components in the i^{th} coordinate directions, and u_1 denotes the axial velocity.

Equation (14) has been discussed in detail by Savins (*28*) and by Astarita and Nicodemo (*3*) and may be useful, in fact, for measurement of the τ_{ii} (that is, the rheological properties) of the fluid. Clearly one does not, however, obtain velocity information unless these terms are first known, and then only by taking differences between the several terms involved. If the static (radial) pressure reading is taken at the Pitot also, rather than at the wall, the integral vanishes but the first and last terms remain.

2. If the flow conditions are turbulent, an additional contribution due to time averaging of the fluctuating stresses arises in the case of fluids described with nonlinear constitutive equations. This effect has also been studied by Astarita and Nicodemo (*3*).

3. The boundary-layer considerations of the previous part of this paper suggest values of δ_o (Figure 1), the boundary-layer thickness at the leading edge of the probe, which are comparable in magnitude with the Pitot tube diameters used by Astarita and Nicodemo (*3*), Bogue and Metzner (*6*), Eissenberg and Bogue (*9*), and Elata et al. (*10*) and are much larger than the probe diameter employed by Ernst (*11*). In the turbulent core of a flow field, in which the probe diameter appears to have little effect (*6*), this is perhaps not serious except in very small ducts, but this effect is likely to prohibit the use of such probes in the wall region of a flow; it is important to note that the effect may not be eliminated by extrapolation to zero probe diameter, as any effects of the finite boundary-layer thickness may persist.

It is clear from the above that the several available sets of turbulent velocity profile data reported in the literature (*10, 11, 24, 32*) in which none of the above effects were considered are clearly suspect and probably in error, as are any analyses based upon such data. Astarita and Nicodemo (*3*) considered all but the third of the above difficulties; as their measurements did not include the wall region, this last has probably not led to any significant errors in their work.

TRACER PARTICLE STUDIES: LIMITATIONS IMPOSED BY THE UEBLER EFFECT

In view of the very tedious nature of experiments in which local fluid velocities are inferred by photographic or other measurements of the motion of particles, drops, or bubbles suspended in a liquid, this technique, although well developed and long known, has not been widely used. On the other hand, in view of the above limitations of the more usual probes, the use of tracer particles may appear to be of rather direct interest.

In studies of the laminar flow of viscoelastic fluids in a moderately rapidly accelerating velocity field (flow into a tube from an essentially infinite reservoir) it has been possible to show that small tracer bubbles (diameter $\simeq$ 0.05 cm.) appear to follow the fluid motion faithfully, as evidenced by a consistent agreement between the integrated velocity profiles and separately measured volumetric flow rates (*19*). In the case of larger bubbles, however, this was not the situation, and an analysis of the problem reveals that whether or not a given bubble or particle will follow the velocity of the field fluid depends (in addition to considerations based on buoyancy and inertial effects) on whether or not the rates of fluid stretching change appreciably over the radius of the bubble. In viscoelastic fluids large net forces [Equation (10)] may be exerted on the tracer object whenever this is the case. Since high rates of extension of fluid elements (that is, rapid stretch rates) appear to be synonymous with turbulence (*4, 27, 29, 35*), it seems probable that tracer particles will not follow the flows being observed unless the particle or bubble diameter is much smaller than the scale of the flow being observed, and the stretch rates of the fluid do not change significantly over the particle diameter.

These considerations do not appear serious if time-averaged quantities are desired, but they impose what appears to be a serious restriction on the use of suspended tracers to obtain instantaneous fluid velocities in turbulent fields.

CONCLUDING REMARKS

It is seen that heated films or wires, as velocity sensing devices, may be restricted to use at low fluid velocities if instantaneous velocity measurements are to be obtained; Equations (7) and (12) may be used to estimate these velocity level limitations. At higher velocities boundary layers which are of great thickness as compared with the size of the probe, and having thicknesses which may become independent of fluid velocity, are predicted to occur. These general predictions are supported by experimental results, though the lack of complete agreement between experiment and analysis indicates the need for more of both.

If the above effects, imposed by the maximal values of the Deborah number which may exist in viscoelastic flows, are minimized by the use of very large probes or low flow rates, previous analyses (*7, 38*) indicate the effect of viscoelastic properties to be quite small in most, although not all, external flows. These predictions are supported by the heat transfer measurements of Shah and co-workers (*31*), who studied flow over cylinders, and the mass transfer measurements of Weil (*37*) for flow over spheres. Moderately or highly viscoelastic fluids were used, respectively, in these cases, yet no deviations from the boundary-layer analyses for purely viscous fluids were found to occur.

The use of Pitot tubes or similar devices is restricted to positions further from the solid surface than in the case of Newtonian fluids, by the same considerations. This restriction is not as strong as that on heat transfer probes. In addition, however, these probes measure contributions from both the impact velocity and the normal stress field. As the latter effects are clearly nonnegligible (3, 28), the interpretation of such Pitot tube readings requires careful consideration of the normal stress effects.

The conditions under which tracer particles do not follow the flow (the Uebler effect) have been considered briefly.

In general, the restrictions placed upon the use of velocity-sensing devices in viscoelastic fluids, by the above effects, require careful consideration and suggest the need of much further experimentation and analysis if external flows of viscoelastic fluids are to be fully explored and understood. The present contribution should not be viewed as an ultimate or penultimate analysis of these problems but is instead intended to define the macroscopic considerations which are likely to enter into the necessary more detailed analyses.

ACKNOWLEDGMENT

Professor A. J. Acosta and D. F. James kindly provided copies of their data well in advance of publication. This study has been supported by the Office of Naval Research, U.S. Navy.

NOTATION

a, a_o = proportionality constants in boundary-layer thickness–Reynolds number relationships, Equations (4) and (4*a*). a refers to viscoelastic fluids and a_o to Newtonian fluids
C_p = specific heat
d = tube diameter (external diameter)
h = heat transfer coefficient
k = thermal conductivity
N_{Deb} = Deborah number, θ_{fl}/θ_{Pr}. The relevant process time scale θ_{Pr} is taken as $\sqrt{1/(D\Gamma/Dt)}$
$N_{Deb,x}$ = local value of Deborah number within boundary layer at distance x from leading edge of object
N_{Pr} = Prandtl number, $C_p\mu/k$
p = pressure
r = radial position coordinate
t = time
u_1 = axial velocity
U, U_o = external (fluid field) velocities; U denotes the local and U_o the upstream value
v = magnitude of velocity vector
$x, \Delta x$ = axial distance, axial boundary-layer magnitude (Figure 1)

Greek Letters

Γ, Γ_s = deformation rates. Γ is taken as the square root of the second invariant of the deformation rate tensor (5); Γ_s as the local stretch rate $\partial u_1/\partial x_1$
δ = boundary-layer thickness. δ_o denotes the magnitude of δ at the leading edge of the object (Figure 1)
θ = angular position coordinate, Equations (8) to (13)
θ_{fl} = characteristic (relaxation) time of fluid
θ_{Pr} = characteristic process time. A full discussion is given in reference *22*
μ = viscosity coefficient
ν = kinematic viscosity; $\nu = \mu/\rho$
ρ = fluid density
τ = stress. $\Delta\tau$ denotes the difference $\tau_{11} - \tau_{22}$; τ_{11}, τ_{22}, and τ_{33} denote the total stresses in the directions of the x_1, x_2, and x_3 coordinates, respectively, in a rectangular Cartesian coordinate system. The x_1 direction is always chosen as the direction of fluid motion in simple (one-component) velocity fields

LITERATURE CITED

1. Acosta, A. J., and D. F. James, private communication; data from Ph.D. thesis of D. F. James, in preparation.
2. Astarita, Gianni, *Ind. Eng. Chem. Fundamentals*, submitted for publication.
3. ———, and Luigi Nicodemo, *A.I.Ch.E. J.*, **12**, 478 (1966).
4. Batchelor, G. K., *Proc. Royal Soc.* (London), **A213**, 349 (1952); *Proc. Cambridge Phil. Soc.*, **51**, 361 (1955).
5. Bird, R. B., W. E. Stewart, and E. N. Lightfoot, "Transport Phenomena," p. 137, Wiley, New York (1960).
6. Bøgue, D. C., and A. B. Metzner, *Ind. Eng. Chem. Fundamentals*, **2**, 143 (1963).
7. Denn, M. M., *Chem. Eng. Sci.*, **22**, 395 (1967).
8. Eckert, E. R. G., and R. M. Drake, Jr., "Heat and Mass Transfer," p. 174, McGraw-Hill, New York (1959).
9. Eissenberg, D. M., and D. C. Bogue, *A.I.Ch.E. J.*, **10**, 723 (1964).
10. Elata, C., J. Lehrer, and A. Kahanovitz, *Israel J. Technol.*, **4**, 87 (1966).
11. Ernst, W. D., *A.I.Ch.E. J.*, **12**, 581 (1966).
12. Etter, I., and W. R. Schowalter, *Trans. Soc. Rheol.*, **9**, (2), 351 (1965).
13. Ginn, R. F., and A. B. Metzner, "Proceedings Fourth International Congress on Rheology," p. 583, Interscience, New York (1965).
14. Hinze, J. O., "Turbulence," pp. 76-79, McGraw-Hill, New York (1959).
15. Kapoor, N. N., J. W. Kalb, E. A. Brumm, and A. G. Fredrickson, *Ind. Eng. Chem. Fundamentals*, **4**, 186 (1965).
16. Leathrum, R. A., Chem. Eng. Rept., Univ. Delaware, Newark (1966).
17. Lodge, A. S., "Elastic Liquids," pp. 114-118, Academic Press, New York (1964).
18. Merrill, E. W., private communication (1966).
19. Metzner, A. B., *A.I.Ch.E. J.*, **13**, No. 2, 316 (1967); E. A. Uebler, Ph.D. thesis, Univ. Delaware, Newark (1966).
20. Metzner, A. B., and Martin Johnson, paper presented at 1966 Annual Meeting, Society of Rheology.
21. Metzner, A. B., and J. L. White, *A.I.Ch.E. J.*, **11**, 989 (1965).
22. ————, and M. M. Denn, *Chem. Eng. Progr.*, **62**, No. 12, 81 (1966).
23. ————, *A.I.Ch.E. J.*, **12**, 863 (1966).
24. Meyer, W. A., *ibid.*, **12**, 522 (1966).
25. Oliver, D. R., *Can. J. Chem. Eng.*, **44**, 100 (1966).
26. Pipkin, A. C., *Quart. Appl. Math.*, **23**, 297 (1966).
27. Reid, W. H., *Proc. Cambridge Phil. Soc.*, **51**, 350 (1955).
28. Savins, J. G., *A.I.Ch.E. J.*, **11**, 673 (1965).
29. ———, *Soc. Petrol. Engrs. J.*, **4**, 203 (1964).
30. Seyer, F. A., and A. B. Metzner, "Proceedings Sixth Naval Hydrodynamics Symposium," Office Naval Res., Washington, D. C. (1966).
31. Shah, M. J., E. E. Peterson, and Andreas Acrivos, *A.I.Ch.E. J.*, **8**, 542 (1962).
32. Shaver, R. G., and E. W. Merrill, *ibid.*, **5**, 181 (1959).
33. Shertzer, C. R., Ph.D. thesis, Univ. Delaware, Newark (1965).
34. Spriggs, T. W., J. D. Huppler, and R. B. Bird, *Trans. Soc. Rheol.*, **10**, No. 1, 191 (1966).
35. Taylor, G. I., *Proc. Royal Soc.* (*London*), **A164**, 15 (1938).
36. Tokita, N., and J. L. White, *J. Appl. Polymer Sci.*, **10**, 1011 (1966).
37. Weil, C., Chem. Eng. Rept., Univ. Delaware, Newark (1966).
38. White, J. L., and A. B. Metzner, *A.I.Ch.E. J.*, **11**, 324 (1965).
39. ————, *J. Appl. Polymer Sci.*, **7**, 1867 (1963).

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